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LETTER TO THE EDITOR

Lattice decorations and one-dimensional percolation

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Abstract. It is well known that in long range one-dimensional percolation the critical exponents depend on the range. We discuss a family of decorations which have range-dependent exponents in one dimension, although the corresponding decorations in higher dimension preserve the values of the exponents. These decorations clarify the unusual nature of one-dimensional percolation.

One-dimensional percolation is of interest primarily for two reasons. Firstly, several one-dimensional problems are exactly soluble and secondly, exponents are found to be range dependent (Klein *et al* 1978, Shalitin 1981, Zhang *et al* 1983, Li *et al* 1983, Schulman 1983). This result was unexpected in the light of the universality hypothesis which, loosely speaking, suggests that exponents should depend only on the dimension of a lattice, and not on its detailed structure. In dimensions greater than one, the universality hypothesis is convincingly supported through numerical evidence (see, e.g., Gaunt and Sykes 1983) and through contact with thermal critical phenomena (Kasteleyn and Fortuin 1969). However, by phrasing the one-dimensional many-neighbour site problem in spin language, Klein *et al* pointed out that the many-neighbour percolation problem mapped on to an Ising-like problem with multi-spin interactions. Such multi-spin interactions have been known to change exponents (Baxter and Wu 1973). However, it is not clear why such interactions affect one-dimensional exponents but not exponents in higher dimensions.

Although the universality hypothesis remains unproven for any pair of unrelated lattices, the hypothesis has been verified on families of lattices related by particular forms of bond or site decoration (Ord *et al* 1984, Ord and Whittington 1985). In these papers, it was found that the exponents were invariant under decoration provided that the critical threshold p_c of the parent lattice was in the *open* interval, $p_c \in (0, 1)$. It was also found that if $p_c = 1$ the exponents $(2-\alpha)$, β, γ, δ and ν *could* be changed, through site decoration, by integer multiples. In this letter we shall show that the decoration families in which exponents are range independent in higher dimensions, have range-dependent exponents in one dimension due to the triviality of the percolation threshold.

The one-dimensional many-neighbour site problem has been solved by the generating function technique (see Reynolds *et al* 1977, Klein *et al* 1978). We illustrate the generating function technique for the one-dimensional first-neighbour problem as follows. We occupy sites with density p and 'ghost' sites with density h. The mean number of clusters of size s, not connected to any ghost site, normalised per occupied site is

$$n_s(p, h) = (1-p)^2 p^{s-1}(1-h)^s, \qquad s \geqslant 1.$$

The mean number of clusters per occupied site is then just

$$G_1(p,h)=\sum_{s\geq 1} n_s(p,h)$$
$$=q^2\sum_{s\geq 1} p^{s-1}(1-h)^s$$
$$=q^2\frac{(1-h)}{[1-p(1-h)]} \qquad (1)$$

where $q=1-p$.

This generalises to l nearest neighbours as (Klein *et al* 1978)

$$G_l(p,h)=\frac{(q^l)^2(1-h)}{1-(1-q^l)(1-h)}. \qquad (2)$$

We note that

$$G_l(p,h)=G_1(f_l(p),h) \qquad (3)$$

where

$$f_l(p)=1-(1-p)^l \qquad (4)$$

From (2) we see that $p_c=1$ for all l and we note that in the neighbourhood of $q=h=0$

$$G_l(p,h)\sim\frac{q^{2l}}{q^l+h} \qquad (5)$$

so that with scaling fields q and h, the scaling powers of q and h are respectively

$$a_q=l^{-1} \qquad a_h=1 \qquad (6)$$

and (Hankey and Stanley 1972)

$$\beta=0 \qquad \delta=\infty \qquad \gamma=l \qquad 2-\alpha=l. \qquad (7)$$

Using hyperscaling, or through direct calculation, one also has

$$\nu=l. \qquad (8)$$

An alternative and instructive way to approach this type of many-neighbour problem is through site decorations (Ord and Whittington 1985).

For a lattice L, we create an *n-pole decoration* L^n of the lattice by replacing each site on L with a complete graph on n vertices (an n-pole), and each bond on L by the additional bonds needed to convert two neighbouring n-poles into a complete graph on $2n$ vertices.

To make contact with many-neighbour percolation in one dimension we let L be the one-dimensional lattice. We define a lattice $\mathscr{L}^n$ and number the sites on this lattice by $k=ni+j$, where i numbers the n-poles and j numbers the sites in an n-pole, $1\leq j\leq n$, in L^n. $\mathscr{L}^n$ is topologically identical to L^n and is a one-dimensional lattice with isotropic many-neighbour interactions up to n sites, and anisotropic interactions up to $2n-1$ sites. In figure 1 we give an example in which $n=2$.

We define a mapping from configurations on L^n to configurations on L such that a site is occupied on L if and only if at least one site is occupied in the corresponding n-pole on L^n. A cluster is the section graph of L or of L^n induced by the occupied sites.

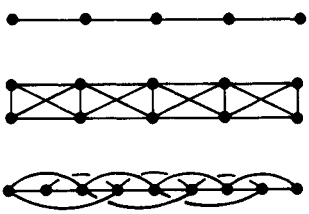

Figure 1. A one-dimensional lattice L, two-pole decoration L^2 and the topologically equivalent many-neighbour lattice $\mathscr{L}^2$.

If sites on L^n are occupied uniformly and independently, with probability p, then the induced site density on L is given by

$$f_n(p) = 1-(1-p)^n.$$

Quantities such as the mean number of clusters $K(p)$, percolation probability $P(p)$ and the mean size of finite clusters $S(p)$, can be related on L^n and L by functional composition with f. It has been shown that since $f(p)$ is continuously differentiable, the critical exponents are invariant under such decorations, provided that the critical density p_c on L is in the open interval (0, 1) (Ord *et al* 1984, Ord and Whittington 1985).

To obtain exponents for the many-neighbour one-dimensional lattices $\mathscr{L}^n$ we calculate the generating function $\tilde{G}_n(p, h)$ on the decorated lattice. We note that

$$\tilde{n}_s^j(p, h) = q^{2n}p^{s-1}q^{nj-s}(1-h)^s C_j^s/n \tag{9}$$

where $\tilde{n}_s^j$ is the number of clusters of size s spanning j poles, q^{2n} accounts for the two perimeter empty poles, q^{nj-s} accounts for the empty sites within the pole cluster, and C_j^s is the number of ways of distributing s sites in j n-poles such that no pole is empty. The factor of n accounts for normalisation per site. We note that the generating function of the C_j^s is

$$\begin{aligned} F_j(s) &= \left(nx + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n\right)^j \\ &= \sum_{k=j}^{nj} C_j^k x^k \\ &= [(1+x)^n - 1]^j. \end{aligned} \tag{10}$$

Thus

$$\begin{aligned} \tilde{G}_l(p, h) &= \sum_{s=1}^{\infty} \sum_{j=[s/n]}^{s} \tilde{n}_s^j(p, h) \\ &= \sum_{j=1}^{\infty} \sum_{s=1}^{nj} q^{2n}p^{s-1}q^{nj-s}C_j^s(1-h)^s/n \\ &= \sum_{j=1}^{\infty} q^{nj}p^{-1} \sum_{s=j}^{nj} C_j^s[p(1-h)/q]^s q^{2n}/n \\ &= \sum_{j=1}^{\infty} q^{nj}p^{-1}F_j[p(1-h)/q]q^{2n}/n \\ &= \sum_{j=1}^{\infty} p^{-1}q^{2n}[(1-ph)^n - q^n]^j/n. \end{aligned} \tag{11}$$

For $h \ll 1$ this may be written as

$$\tilde{G}_l(p, h) = p^{-1} \sum_{j=1}^{\infty} q^{2n}(1-q^n)^j \left(1 - \frac{nph}{1-q^n}\right)^j n^{-1}$$
$$= f_n(p)p^{-1} G\left(f_n(p), \frac{np}{f_n(p)} h\right) n^{-1}. \tag{12}$$

Thus we see that the n nearest-neighbour problem and the n-pole decoration are related to the single-neighbour problem by functional composition of *the same* density function f_n. The difference in the two problems occurs only in the appearance of the scalar field $hnp/f_n(p)$ and the multiplicative prefactor in (12). These differences do not affect the asymptotic behaviour since near $p = 1$, $h = 0$:

$$\tilde{G}_l(p, h) \sim q^{2n}/(q^n + nh) \tag{13}$$

and we see that the decorated lattice has the same exponents as those in (7) and (8).

Thus site decorations, which preserve exponents in higher dimension, do not do so in one dimension. The changing of the exponents by a factor of n is a direct result of the vanishing of the first $n-1$ derivations of $f_n(p)$ at p_c (see Ord and Whittington 1985). In one dimension, the number of derivatives which vanish depends on the range of interaction in both the isotropic and anisotropic cases. In higher dimensions, where in general $p_c \neq 1$, the first derivative is non-zero in the anisotropic case, so that the exponents are range independent. This strongly suggests that exponents are also range independent in the isotropic case.

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